*Journal of Global Optimization* **27:** 411–426, 2003.



# Gap Functions for Equilibrium Problems

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(Received 28 March 2002; accepted in revised form 11 March 2003)

**Abstract.** The theory of gap functions, developed in the literature for variational inequalities, is extended to a general equilibrium problem. Descent methods, with exact an inexact line-search rules, are proposed. It is shown that these methods are a generalization of the gap function algorithms for variational inequalities and optimization problems.

**Key words:** descent methods, equilibrium problems, gap functions, variational inequalites.

## 1. Introduction

The gap function approach, which has widely been studied for variational inequalities (for short, $VI$), can be extended to an equilibrium problem (for short, $EP$):

$$\text{find } y^* \in K \text{ s.t. } f(x, y^*) \geq 0, \quad \forall x \in K, \tag{$EP$}$$

where $f : X \times X \longrightarrow \mathbb{R}$, with $f(x, x) = 0$, for all $x \in K$, and $K$ is a convex subset of the set $X \subseteq \mathbb{R}^n$.

It is well known (see e.g. [3]) that $(EP)$ provides a general setting which includes several problems as $VI$, complementarity problems, optimization problems, etc. For example, if we define $f(x, y) := \langle F(y), x - y\rangle$ then $EP$ collapses into the classic $VI$:

$$\text{find } y^* \in K \text{ s.t. } \langle F(y^*), x - y^*\rangle \geq 0, \quad \forall x \in K, \tag{$VI$}$$

where $F : X \longrightarrow \mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$. If $f(x, y) := J(x) - J(y)$ then $EP$ is equivalent to the optimization problem

$$\min_{x\in K} J(x) \quad \text{s.t.} \quad x \in K,$$

where $J : X \longrightarrow \mathbb{R}$. We refer to [3] and references therein for an exhaustive survey concerning the main applications and existence results of a solution of the problem $EP$.

We start our analysis by proving that $(EP)$ is equivalent to the minimax problem

$$\min_{y\in K} \sup_{x\in K} [-f(x, y)], \tag{1}$$

provided that the optimal value in (1) is zero; this leads to consider the function

$$g(y) = \sup_{x \in K} [-f(x, y)] \tag{2}$$

whose minimization on the set $K$ coincides with the problem (1). Peculiar properties of the function $g$ are the non-negativity on the set $K$ and the fact that $g(y^*) = 0$ if and only if $y^*$ is a solution of $EP$. The functions which fulfil the above mentioned properties form the class of the gap functions associated to $EP$. The function (2), that in general is not differentiable, has been analysed in the case of a variational inequality problem; Zhu and Marcotte [22] proved that

$$g(y) := \max_{x \in K} [\langle F(y), y - x \rangle - H(x, y)], \tag{3}$$

is a continuously differentiable gap function for $VI$ under the following conditions:

$H(x, y) : X \times X \longrightarrow \mathbb{R}$, is a non–negative, continuously differentiable, strongly convex function on the convex set $K$ with respect to $x$, such that

$$H(y, y) = 0 \quad \text{and} \quad H'_x(y, y) = 0, \quad \forall y \in K.$$

In the particular case where $H(x, y) := \frac{1}{2}\langle x - y, M(x - y)\rangle$, with $M$ symmetric and positive definite matrix of order $n$, it is recovered the gap function introduced by Fukushima [7].

It is shown [13] that the results obtained in [22, 7] are closely related to the introduction of an auxiliary $VI$. Following the line developed in [13, 14], where it is proved that $EP$ is equivalent to the auxiliary equilibrium problem (for short, $AEP$):

$$\text{find } y^* \in K \text{ s.t. } f(x, y^*) + H(x, y^*) \geq 0 \quad \forall x \in K,$$

we will show that the minimax formulation of $AEP$ allows us to define a continuously differentiable gap function for $EP$ that collapses into (3) when $EP$ represents the problem $VI$. A direct consequence of the analysis is the definition of line search algorithms for the solution of $EP$ based on the minimization of suitable gap

functions. These algorithms are a generalization of those proposed by Fukushima for $VI$ [7]. The analysis of the gap function approach for $EP$ allows to extend the applications to further variational formulations as the Minty Variational Inequality [8]:

$$\text{find } y^* \in K \text{ s.t. } \langle F(x), x - y^* \rangle \geq 0, \quad \forall x \in K \tag{MVI}$$

which, under the hypotheses of continuity and pseudomonotonicity of the operator $F$, is equivalent to $VI$ [12].

$MVI$ can be obtained as an $EP$ defining $f(x, y) := \langle F(x), x - y \rangle$.
We recall the main notations and definitions that will be used in the sequel.

Let $Y \subseteq \mathbb{R}^n$. A point to set map $A : X \longrightarrow 2^Y$ is upper semicontinuous (u.s.c.) according to Berge at a point $\lambda^* \in X$ if, for each open set $B$ containing $A\lambda^*$, there exists a neighborhood V of $\lambda^*$ such that

$$A\lambda \subset B, \quad \forall \lambda \in V.$$

A function $f : X \times X \longrightarrow \mathbb{R}$ is said strongly monotone on $K \subseteq X$, with modulus $a > 0$, iff:

$$f(x, y) + f(y, x) \leq -a\|y - x\|^2, \quad \forall x, y \in K.$$

When $f$ is differentiable with respect to $x$ (resp. to $y$), we will denote by $f'_x$ (resp. $f'_y$) the gradient of $f$ with respect to $x$ (resp. to $y$).

A function $h : X \longrightarrow \mathbb{R}$ is said strongly convex on $K$ with modulus $a > 0$ iff, $\forall x_1, x_2 \in K$ and $\forall \lambda \in [0, 1]$,

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda h(x_1) + (1 - \lambda)h(x_2) - a[\lambda(1 - \lambda)/2]\|x_1 - x_2\|^2.$$

We will say that the mapping $F : X \longrightarrow \mathbb{R}^n$ is monotone on $K \subseteq X$ iff:

$$\langle F(y) - F(x), y - x\rangle \geq 0, \quad \forall x, y \in K;$$

it is strictly monotone if strict inequality holds $\forall x \neq y$.
We will say that the mapping $F$ is strongly monotone, with modulus $a > 0$, on $K$ iff:

$$\langle F(y) - F(x), y - x\rangle \geq a\|y - x\|^2, \quad \forall x, y \in K;$$

$F$ is Lipschitz continuous with modulus $L > 0$ over $K$ iff

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in K.$$

If $f : X \longrightarrow \mathbb{R}$ is continuously differentiable and if $f'$ is Lipschitz continuous on $K$, with modulus $L$, then we have:

$$f(x) - f(y) \leq \langle f'(y), x - y\rangle + (L/2)\|x - y\|^2, \quad \forall x, y \in K.$$

## 2. Gap functions for equilibrium problems

The starting point of our analysis is the introduction of suitable equivalent formulations of the equilibrium problem: the first one is the classic minimax formulation of $EP$, while the second is a regularization of $EP$ obtained by adding to the function $f(x, y)$ the additional term $H(x, y)$.

The following preliminary result (see, e.g., [13]) states the minimax formulation of $EP$.

LEMMA 2.1 *Suppose that $f(x, x) = 0$, $\forall x \in K$. Then, the following statements are equivalent:*
*i) there exists $y^* \in K$ s.t. $f(x, y^*) \geq 0$, $\forall x \in K$.*
*ii)* $\min_{y \in K} \sup_{x \in K} [-f(x, y)] = 0$.

The previous formulation leads to the introduction of the gap function associated to the problem $EP$, a natural extension of the one considered for variational inequalities [1].

DEFINITION 2.1 *Let $K \subseteq X$. The function $p : X \longrightarrow \mathbb{R}$ is a gap function for $EP$ iff:*
*i) $p(y) \geq 0$, $\forall y \in K$;*
*ii) $p(y) = 0$ and $y \in K$ iff $y$ is a solution for $EP$.*

It is immediate to observe that

$$g(y) := \sup_{x \in K} [-f(x, y)] \tag{4}$$

is a gap function for $EP$.
When $EP$ represents the problem $VI$ we recover the gap function

$$p(y) := \sup_{x \in K} \langle F(y), y - x \rangle,$$

introduced by Auslender [1]. This function, in general, is not differentiable; the problem of defining a continuously differentiable gap function for $VI$ was first solved by Fukushima [7] whose approach was generalized by Zhu and Marcotte [22]. We will show that the results obtained in [7, 22] are closely related to the introduction of an auxiliary equilibrium problem that allows to regularize the original $EP$ so that the gap function (4) associated to the $AEP$ is continuously differentiable. First of all, we will state sufficient conditions that guarantee the differentiability of (4).

PROPOSITION 2.1 *Assume that the following conditions hold:*
*i) $f(x, y)$ is a strictly convex function with respect to $x$, for every $y \in K$;*
*2i) $f$ is differentiable with respect to $y$, for every $x \in K$ and $f'_y$ is continuous on $K \times K$;*
*3i) the supremum in (4) is attained for every $y \in K$. Then $g(y) := \sup_{x \in K} [-f(x, y)]$ is a continuously differentiable gap function for $EP$ and its gradient is given by*

$$g'(y) = -f'_y(x(y), y), \tag{5}$$

*where $x(y) := argmin_{x \in K} f(x, y)$.*

*Proof.* Since $f(x, y)$ is strictly convex with respect to $x$ then there exists a unique minimum point $x(y)$ of the problem $\min_{x \in K} f(x, y)$. Applying Theorem 4.3.3 of [2], we obtain that $x(y)$ is u.s.c. according to Berge at $y$ and, being $x(y)$ single-valued, it follows that it is continuous at $y$.
Since $f'_y$ is continuous, from theorem 1.7, chapter 4 of [1], it follows that

$$g'(y) = -f'_y(x(y), y).$$

From the continuity of $f'_y$ and $x(y)$ it follows that $g'(y)$ is continuous at $y$. □

REMARK 2.1 Obviously (3i) is fulfilled if $f(x, y)$ is l.s.c. with respect to $x$ and the set $K$ is compact, or if $f(\cdot, y)$ is strongly convex. Proposition 2.1 is still valid if $f(\cdot, y)$ is assumed to be strictly quasi-convex [19].

The hypothesis of strict convexity for the function $f(\cdot, y)$ is not always fulfilled: for example, for the problem $VI$, $f(\cdot, y)$ is linear. We can overcome this difficulty by introducing an auxiliary equilibrium problem, adding to the operator $f$ a strictly convex term $H(\cdot, y)$.

Let $H(x, y) : X \times X \longrightarrow \mathbb{R}$ be a differentiable function on $K$ with respect to $x$, and such that:

$$H(x, y) \geq 0, \quad \forall (x, y) \in K \times K; \tag{6}$$

$$H(y, y) = 0, \quad \forall y \in K; \tag{7}$$

$$H'_x(y, y) = 0, \quad \forall y \in K. \tag{8}$$

The auxiliary equilibrium problem ($AEP$) consists in finding $y^* \in K$ such that

$$f(x, y^*) + H(x, y^*) \geq 0, \quad \forall x \in K.$$

The following result proves the equivalence between $EP$ and $AEP$.

PROPOSITION 2.2 *[14] Let $f(x, y)$ be a convex differentiable function with respect to $x$, $\forall y \in K$. Then $y^*$ is a solution of $EP$ iff it is a solution of $AEP$.*

Therefore we can apply Proposition 2.1 to $AEP$ in order to obtain a continuously differentiable gap function for $EP$.

THEOREM 2.1 *Suppose that $K$ is a closed subset in $X$, $f(x, y)$ is a differentiable l.s.c. convex function with respect to $x$, for every $y \in K$, differentiable with respect to $y$ and that $f'_y$ is continuous on $K \times K$. Let $H(x, y) : X \times X \longrightarrow \mathbb{R}$ be a continuously differentiable function on $K \times K$, strongly convex with respect to $x$, for every $y \in K$, and such that (6), (7) and (8) hold. Then*

$$h(y) := \max_{x \in K}[-f(x, y) - H(x, y)] \tag{9}$$

*is a continuously differentiable gap function for $EP$ and its gradient is given by*

$$h'(y) = -f'_y(x(y), y) - H'_y(x(y), y),$$

*where* $x(y) := argmin_{x \in K}[f(x, y) + H(x, y)]$.

*Proof.* By Proposition 2.2 we obtain that $EP$ is equivalent to $AEP$. Applying Proposition 2.1 to the auxiliary problem $AEP$, we complete the proof. □

REMARK 2.2 When $f(x, y) := \langle F(y), x - y\rangle$ the gap function defined by (9) collapses into the one considered by Zhu and Marcotte [22] for $VI$.
When $f(x, y) := \langle F(x), x - y\rangle$ we obtain a gap function associated to $MVI$ (see [13]). Furthermore, we observe that, if $K$ is a compact set, then $H(\cdot, y)$ may be assumed to be strictly convex.

## 3. Descent methods for EP

The gap function approach coupled with the auxiliary problem principle allows to express $EP$ by means of the constrained minimization of a continuously differentiable function. This makes possible to consider exact and inexact line-search algorithms in order to minimize (4) or (9). If not differently specified, in this section we will consider the following assumptions:

i) $K$ is a convex set in $X$;
2i) $f(x, y)$ is a differentiable convex function with respect to $x$, for every $y \in K$;
3i) $f(x, y)$ is differentiable on $K$ with respect to $y$, for every $x \in K$;
4i) $f'_y$ is continuous on $K \times K$.

**Algorithm 3.1**

Let $g$ be defined by (4).
Step 1. Let $k = 0$, $y_0 \in K$;
Step 2. $y_{k+1} := y_k + t_k d_k$,
where $d_k := x(y_k) - y_k$ , $x(y_k)$ is the solution of the problem:

$$\min_{x \in K} f(x, y_k), \tag{10}$$

and $t_k$ is the solution of the problem

$$\min_{0 \le t \le 1} g(y_k + t d_k).$$

Step 3. If $\|y_{k+1} - y_k\| < \mu$, for some fixed $\mu > 0$, then STOP, otherwise put $k = k + 1$ and go to Step 2.

It must be proved that $d_k$ is a descent direction for $g$ at the point $y_k$.
To this aim it is necessary to make the following further assumption:

$$\langle f'_x(x, y) + f'_y(x, y), x - y\rangle \geq 0, \quad \forall (x, y) \in K \times K. \tag{11}$$

REMARK 3.1 If $f(x, y) := \langle F(x), x - y\rangle$ or $f(x, y) := \langle F(y), x - y\rangle$ then (11) holds provided that $\nabla F(x)$ is a positive semidefinite matrix, $\forall x \in K$ (see Section 5). It can also be proved that condition (11) holds if $f$ is convex on $K$ with respect to $x$ and concave on $K$ with respect to $y$.

PROPOSITION 3.1 *Suppose that the hypotheses of the Proposition 2.1 hold and moreover the assumption (11) is fulfilled. Then $d(y) := x(y) - y$ is a descent direction for $g$ at $y \in K$, provided that $x(y) \neq y$.*

*Proof.* We preliminarly observe that $y^*$ is a solution for $EP$ iff $x(y^*) = y^*$. Since $x(y) := \arg\min_{x\in K} f(x, y)$ and $f(\cdot, y)$ is strictly convex the following variational inequality holds:

$$\langle f'_x(x(y), y), z - x(y)\rangle > 0, \quad \forall z \in K, z \neq x(y). \tag{12}$$

Putting $z := y$ we obtain $\langle f'_x(x(y), y), x(y) - y\rangle < 0$.
Taking into account (11) we have

$$0 \; > \; \langle f'_x(x(y), y), x(y) - y\rangle \; \geq \; -\langle f'_y(x(y), y), x(y) - y\rangle.$$

By (5) we get $\langle g'(y), x(y) - y\rangle < 0$. □

THEOREM 3.1 *Suppose that $K$ is a compact set in $X$, the assumption (11) is fulfilled and $f(x, y)$ is a strictly convex function with respect to $x$, for every $y \in K$.*

*Then, for any $y_0 \in K$ the sequence $\{y_k\}$ defined by Algorithm 3.1, belongs to set $K$ and any accumulation point of $\{y_k\}$ is a solution of $EP$.*

*Proof.* The convexity of $K$ implies that the sequence $\{y_k\} \subset K$ since $t_k \in [0, 1]$. Since $x(y)$ is continuous (see the proof of the Proposition 2.2) the function $d(x) := x(y) - y$ is continuous on $K$. It is known (see e.g.[15]) that the map

$$U(y, d) := \{x : x = y + t_k d, \; g(y + t_k d) = \min_{0\leq t\leq 1} g(y + td)\}$$

is closed whenever $g$ is a continuous function. Therefore the algorithmic map $y_{k+1} = U(y_k, d(y_k))$ is closed, (see e.g. [15]). Zangwill's convergence theorem [21] implies that any accumulation point of the sequence $\{y_k\}$ is a solution of $EP$.
□

A further algorithm can be obtained applying the Algorithm 3.1 to the auxiliary equilibrium problem $AEP$.

**Algorithm 3.2**

Let $h$ be defined by (9).

Step 1. Let $k = 0$, $y_0 \in K$;
Step 2. $y_{k+1} := y_k + t_k d_k$, $k = 1, \ldots$
where $d_k := x(y_k) - y_k$ , $x(y_k)$ is the solution of the problem:

$$\min_{x \in K} [f(x, y_k) + H(x, y_k)],$$

and $t_k$ is the solution of the problem

$$\min_{0 \leq t \leq 1} h(y_k + t d_k).$$

Step 3. If $\|y_{k+1} - y_k\| < \mu$, for some fixed $\mu > 0$, then STOP, otherwise put $k = k + 1$ and go to Step 2.

In order to apply Algorithm 3.2 we must replace (11) with the condition

$$\langle f_x'(x, y) + H_x'(x, y) + f_y'(x, y) + H_y'(x, y), x - y \rangle \geq 0, \quad \forall (x, y) \in K \times K. \tag{13}$$

If we make the assumption ( also considered in [20, 13]):

$$H_x'(x, y) + H_y'(x, y) = 0, \quad \forall (x, y) \in K \times K, \tag{14}$$

then (13) obviously collapses to (11).
We observe that assumption (14) is fulfilled in the exact line search algorithm proposed by Fukushima in [7] which can be obtained putting $H(x, y) = \frac{1}{2}\langle M(x - y), x - y \rangle$, where $M$ is a symmetric positive definite matrix of order $n$.

The convergence of Algorithm 3.2 is a direct consequence of Theorem 3.1.

THEOREM 3.2 *Let $H(x, y) : X \times X \longrightarrow \mathbb{R}$ be a continuously differentiable function on $K \times K$, strictly convex with respect to $x$, for every $y \in K$, and such that (6), (7) and (8) hold. Suppose that $K$ is a compact set in $X$ and that the assumption (13) is fulfilled.*

*Then, for any $y_0 \in K$ the sequence $\{y_k\}$ defined by Algorithm 3.2, belongs to the set $K$ and any accumulation point of $\{y_k\}$ is a solution of $EP$.*

*Proof.* By Proposition 2.2 we obtain that $EP$ is equivalent to $AEP$. Applying Theorem 3.1 to the auxiliary problem $AEP$ we complete the proof. $\square$

Algorithm 3.1 is based on an exact line-search rule: it is possible to consider the inexact version of the previous method. To this end, we must introduce a generalization of condition (11):

$$\langle f'_x(x, y) + f'_y(x, y), x - y\rangle \geq \mu\|x - y\|^2, \quad \forall (x, y) \in K \times K, \tag{15}$$

where $\mu > 0$ is a suitable constant.

REMARK 3.2 It is possible to prove that (15) is fulfilled if $f(x, y)$ is strongly convex with respect to $x$, with modulus $2\mu$, for every $y \in K$, and concave, with respect to $y$, for every $x \in K$.

PROPOSITION 3.2 *Suppose that $K$ is a compact set in $X$ and the assumption (15) is fulfilled for a suitable $\mu > 0$. Then*

$$\langle g'(y), d(y)\rangle \leq -\mu\|d(y)\|^2,$$

*where $d(y) := x(y) - y$.*

*Proof.* Following the proof of Proposition 3.1, we obtain

$$0 \geq \langle f'_x(x(y), y), x(y) - y\rangle \geq -\langle f'_y(x(y), y), x(y) - y\rangle + \mu\|x(y) - y\|^2.$$

By (5) we get $\langle g'(y), d(y)\rangle \leq -\mu\|d(y)\|^2$. □

**Algorithm 3.3.**

Step 1. Let $y_0$ be a feasible point, $\epsilon$ be a tolerance factor and $\beta, \sigma$ parameters in the open interval $(0, 1)$. Let $k = 0$.

Step 2. If $g(y_k) = 0$, then STOP, otherwise go to step 3.

Step 3. Let $d_k(y_k) := x(y_k) - y_k$. Select the smallest nonnegative integer $m$ such that

$$g(y_k) - g(y_k + \beta^m d_k) \geq \sigma\beta^m\|d_k\|^2,$$

set $\alpha_k = \beta^m$ and $y_{k+1} = y_k + \alpha_k d_k$.
If $\|y_{k+1} - y_k\| < \epsilon$, then STOP, otherwise let $k = k + 1$ and go to step 2.

THEOREM 3.3 *Let $\{y_k\}$ be the sequence defined in the Algorithm 3.3. Suppose that $K$ is a compact set in $X$, $f(x, y)$ is a strictly convex function with respect to $x$, for every $y \in K$, and the assumption (15) is fulfilled for a suitable $\mu > 0$ and $\sigma < \mu/2$.*

*Then, for any $x_0 \in K$ the sequence $\{y_k\} \subset K$ and any accumulation point of $\{y_k\}$ is a solution of $EP$.*

*Proof.* The convexity of $K$ implies that the sequence $\{y_k\} \subset K$, since $\alpha_k \in [0, 1]$. The compactness of $K$ ensures that $\{y_k\}$ has at least one accumulation point. Let $\{\tilde{y}_k\}$ be any convergent subsequence of $\{y_k\}$ and $y^*$ be its limit point.
We will prove that $x(y^*) = y^*$ so that $y^*$ is a solution of $EP$.

Let $d(y) := x(y) - y$; since $x(y)$ is continuous (see the proof of Proposition 2.1) it follows that $d(y)$ is continuous; therefore we obtain that $d(\tilde{y}_k) \to d(y^*) := d^*$ and $g(\tilde{y}_k) \to g(y^*) := g^*$. By the line search rule we have

$$g(y_k) - g(y_{k+1}) \geq \sigma \alpha_k \|d_k\|^2,$$

and this relation remains valid for the subsequence $\{\tilde{y}_k\}$. Therefore,

$$\tilde{\alpha}_k \|d(\tilde{y}_k)\|^2 \to 0,$$

for a suitable subsequence $\{\tilde{\alpha}_k\} \subseteq \{\alpha_k\}$.

If $\tilde{\alpha}_k > \gamma > 0$, $\gamma \in \mathbb{R}$, $\forall k \in N$, then $\|d(\tilde{y}_k)\| \to 0$ so that $x(y^*) = y^*$.
Otherwise suppose that there exists a subsequence $\{\alpha_{k'}\} \subseteq \{\tilde{\alpha}_k\}$, $\alpha_{k'} \longrightarrow 0$. By the line search rule we have that

$$\frac{g(\tilde{y}_{k'}) - g(\tilde{y}_{k'} + \bar{\alpha}_{k'} d(\tilde{y}_{k'}))}{\bar{\alpha}_{k'}} < \sigma \|d(\tilde{y}_{k'})\|^2, \tag{16}$$

where $\bar{\alpha}_{k'} = \alpha_{k'}/\beta$.
Taking the limit in (16) for $k' \to \infty$, since $\bar{\alpha}_{k'} \to 0$ and $g$ is continuously differentiable, we obtain

$$-\langle g'(y^*), d^* \rangle \leq \sigma \|d^*\|^2. \tag{17}$$

Recalling Proposition 3.2, we have also

$$-\langle g'(y^*), d^* \rangle \geq \mu \|d^*\|^2.$$

Since $\sigma < \mu/2$, it must be $\|d^*\| = 0$, which implies $x(y^*) = y^*$. □

The algorithm 3.3 can also be applied for the inexact minimization of the gap function $h$ defined by (9), provided that the assumption (15) is replaced by:

$$\langle f'_x(x, y) + H'_x(x, y) + f'_y(x, y) + H'_y(x, y), x - y \rangle \geq \mu \|x - y\|^2, \tag{18}$$
$$\forall (x, y) \in K \times K,$$

We obtain the following method, appliable also in the case where $f(\cdot, y)$ is convex but not necessarily strictly convex.

**Algorithm 3.4.**

Step 1. Let $y_0$ be a feasible point, $\epsilon$ be a tolerance factor and $\beta, \sigma$ parameters in the open interval $(0, 1)$. Let $k = 0$.

Step 2. If $h(y_k) = 0$, then STOP, otherwise go to step 3.

Step 3. Let $d_k(y_k) := x(y_k) - y_k$. Select the smallest nonnegative integer $m$ such that

$$h(y_k) - h(y_k + \beta^m d_k) \geq \sigma\beta^m \|d_k\|^2,$$

set $\alpha_k = \beta^m$ and $y_{k+1} = y_k + \alpha_k d_k$.
If $\|y_{k+1} - y_k\| < \epsilon$, then STOP, otherwise let $k = k + 1$ and go to step 2.

COROLLARY 3.1 *Let $\{y_k\}$ be the sequence defined in the Algorithm 3.4. Suppose that $K$ is a compact set in $X$ and the assumption (18) is fulfilled for a suitable $\mu > 0$ and $\sigma < \mu/2$.*

*Then, for any $x_0 \in K$ the sequence $\{y_k\} \subset K$ and any accumulation point of $\{y_k\}$ is a solution of $EP$.*

In the next section we will prove that it is possible to drop the compactness assumption on the feasible set $K$, in the case where the operator $f$ is strongly monotone and $H'_x(\cdot, y)$ is Lipschitz continuous on $K$.

## 4. Error bounds

In this section we will show that the functions $g$ and $h$ provide a global error bound for $EP$ in the hypothesis of strong monotonicity of the operator $f$.

PROPOSITION 4.1 *Let $f$ be strongly monotone on $K$, with modulus $b$. Then*

$$g(y) \geq b\|y - y^*\|^2, \quad \forall y \in K, \tag{19}$$

*where $y^*$ is the solution of $EP$.*

*Proof.* Since $\forall x \in K$, $g(y) \geq -f(x, y)$ then

$$g(y) \geq -f(y^*, y) - f(y, y^*) + f(y, y^*) \geq$$

$$b\|y - y^*\|^2 + f(y, y^*) \geq b\|y - y^*\|^2.$$

□

In order to extend the previous result to the gap function $h$, we must consider the additional assumption of Lipschitz continuity on $H'_x$.

PROPOSITION 4.2 *Let $f$ be strongly monotone on $K$, with modulus $b$, $H(\cdot, y)$ be convex and $H'_x(\cdot, y)$ Lipschitz continuous with modulus $L < 2b$, for every $y \in K$ . Then*

$$h(y) \geq (b - L/2)\|y - y^*\|^2, \quad \forall y \in K, \tag{20}$$

*where $y^*$ is the solution of $EP$.*

*Proof.* $\forall x, y \in K$, we have

$$h(y) \geq -f(x, y) - H(x, y).$$

Therefore, for $x = y^*$,

$$h(y) \geq -f(y^*, y) - H(y^*, y) - f(y, y^*) + f(y, y^*) \geq$$

$$b\|y - y^*\|^2 + f(y, y^*) - H(y^*, y).$$

We obtain

$$h(y) \geq b\|y - y^*\|^2 - H(y^*, y). \tag{21}$$

Since $H'_x(\cdot, y)$ is Lipschitz continuous, the following inequality holds:

$$H(y^*, y) = H(y^*, y) - H(y, y) \leq (L/2)\|y^* - y\|^2, \quad \forall y \in K.$$

Combining the previous inequality with (21), we get

$$h(y) \geq (b - L/2)\|y - y^*\|^2.$$

□

The hypothesis of strong monotonicity of $f$ allows to drop the compactness assumption on the set $K$, in order to apply the algorithms defined in the previous section. In fact (19) and (20) guarantee that the sequence $\{y_k\}$, generated by any of the above algorithms, is contained in a compact set, taking into account that $\{g(y_k)\}$ is a strictly decreasing sequence.

In the next section, analysing the particular case of a variational inequality, we will see that the strong monotonicity of the operator $F$ is a sufficient condition in order to apply the inexact algorithms 3.3 and 3.4.

## 5. Applications to variational inequalities and optimization problems

In this section we will consider suitable classes of problems that may be solved by means of the gap function algorithms; in particular, we will analyse finite dimensional variational inequalities: this will allow to obtain further applications to optimization problems.

Consider the variational inequality:

$$\text{find } y^* \in K \ \text{ s.t. } \langle F(y^*), x - y^* \rangle \geq \phi(y^*) - \phi(x), \quad \forall x \in K, \quad (VI(F, \phi))$$

where $F : X \longrightarrow \mathbb{R}^n$, $K$ is a convex set in $X$, $\phi : X \longrightarrow \mathbb{R}$.

If we define $f(x, y) := \langle F(y), x - y\rangle - \phi(y) + \phi(x)$ then $EP$ is equivalent to $VI(F, \phi)$.
When the operator $F$ is monotone and $\phi$ is a convex function, $VI(F, \phi)$ is equivalent to the following variational inequality:

$$\text{find } y^* \in K \text{ s.t. } \langle F(x), x - y^*\rangle \geq \phi(y^*) - \phi(x) \quad \forall x \in K. \quad (MVI(F, \phi))$$

When $\phi$ is a constant function on $K$, then $MVI(F, \phi)$ collapses to $MVI$.

PROPOSITION 5.1 *Let $\phi : X \longrightarrow \mathbb{R}$ be a convex function on $K$ and $F$ be a continuous monotone operator on $K$. Then $y^*$ is a solution of $VI(F, \phi)$ iff it is a solution of $MVI(F, \phi)$.*

In order to apply the algorithms, developed in the previous section, to variational inequality problems, we must deepen the analysis of the assumptions (11), (13) and (15).

PROPOSITION 5.2 *Suppose that $F$ is continuously differentiable on $K$, $\phi$ is a differentiable convex function on $K$ and*
*i) $f(x, y) = \langle F(x), x - y\rangle + \phi(x) - \phi(y)$ or*
*ii) $f(x, y) := \langle F(y), x - y\rangle + \phi(x) - \phi(y)$,*
*then*
*1) if $F$ is monotone on $K$, then (11) holds;*
*2) if $F$ is strongly monotone on $K$, with modulus $\mu$, then (15) holds .*

*Proof.* In the case (*i*), we have

$$f_x(x, y) + f_y(x, y) = \nabla F(x)(x - y) + \phi'(x) - \phi'(y),$$

while, in the case (*ii*)

$$f_x(x, y) + f_y(x, y) = \nabla F(y)(x - y) + \phi'(x) - \phi'(y).$$

It is known [18] that $F$ is monotone on $K$ iff $\nabla F(x)$ is a positive semidefinite matrix, $\forall x \in K$, and that $F$ is strongly monotone on $K$, with modulus $\mu$, iff

$$\langle \nabla F(y)d, d\rangle \geq \mu\|d\|^2, \quad \forall d \in \mathbb{R}^n, \forall y \in K.$$

Putting $d := x - y$, we prove the statement. □

We observe that, if $F$ is continuously differentiable and monotone (resp. strongly monotone) on $K$ and $\phi$ is a differentiable strictly convex function on $K$, then Algorithm 3.1 (resp. Algorithm 3.3) can be applied for solving $VI(F, \phi)$.

For the classic $VI$, obtained by putting $\phi(x) = 0, \forall x \in K$, we can apply Algorithm 3.2: actually, since $H(\cdot, y)$ is a strictly convex function, $\forall y \in K$, then, in Theorem 3.1, $f(\cdot, y)$ can be assumed to be convex.

Concerning condition (13), we observe that it is fulfilled when $F$ is monotone on $K$ provided that

$$\langle H'_x(x, y) + H'_y(x, y), x - y \rangle \geq 0, \quad \forall (x, y) \in K \times K.$$

As already observed, if $H$ fulfils the assumption:

$$H'_x(x, y) + H'_y(x, y) = 0, \quad \forall (x, y) \in K \times K, \tag{22}$$

then it is enough to suppose that $F$ is monotone on $K$.

Similarly, the general algorithms can be applied to $MVI(F, \phi)$; Algorithm 3.1 (resp. 3.3) can be applied, in the hypothesis of monotonicity (resp. strong monotonicity) of the operator $F$, with the further additional condition:

$f(x, y) := \langle F(x), x - y \rangle - \phi(y) + \phi(x)$ is a strictly convex function on $K$, with respect to $x$, $\forall y \in K$.

Algorithms 3.2 and 3.4 require the strict convexity (with respect to $x$) of the function

$$f(x, y) := \langle F(x), x - y \rangle + H(x, y) - \phi(y) + \phi(x),$$

so that they can be applied in the case where the function $\langle F(x), x - y \rangle - \phi(y) + \phi(x)$ is convex but not necessarily strictly convex (for example when $F(x) = Ax$, with $A$ positive semidefinite matrix of order $n$ and $\phi(x) = 0$, for $x \in K$).

As regards the applications to optimization problems, it is known (see e.g. [4]) that $VI(\nabla\psi, \phi)$ represents the first order optimality condition of the following constrained extremum problem:

$$\min_{x \in K} [\psi(x) + \phi(x)] \tag{$P$}$$

where $\psi : X \longrightarrow \mathbb{R}$ is a continuously differentiable convex function on the convex set $K$, $\phi : X \longrightarrow \mathbb{R}$ is a continuous strongly convex function on $K$ having finite directional derivative $\phi'(x; x - z)$, $\forall x, z \in K$. For completeness we report the statement that proves the equivalence between $(P)$ and the related variational inequalities.

PROPOSITION 5.3 *$y^*$ is a solution of P iff it a solution of $VI(\nabla\psi, \phi)$ or, equivalently, of $MVI(\nabla\psi, \phi)$.*

*Proof.* See e.g. [1]. □

The previous proposition allows to apply the proposed algorithms to $VI(\nabla\psi, \phi)$ and to $MVI(\nabla\psi, \phi)$ in order to solve the optimization problem $P$, if we assume that $\phi$ is differentiable on $K$.

## 6. Concluding remarks

We have extended the gap function theory to equilibrium problems. This has allowed us to consider descent methods for solving $EP$. We have shown that these methods, that could be stated directly for the original problem, must be applied to an equivalent auxiliary equilibrium problem in order to achieve the convergence under weaker assumptions on the given problem.

Further topics of research in this field are: the connections with the proximal methods for equilibrium problems [6, 17]; the methods based on an unconstrained gap function [20]; the extensions to vector equilibrium problems (see e.g. [10]), the analysis in the image space and the role of the separation techniques [9].

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